JHEP HYPER VERSION

RECEIVED: *January 15, 2001*, ACCEPTED: *January 22, 2001*

Large-N_c QCD meets Regge theory: the example of spin-one two-point functions

Maarten Golterman

Department of Physics, Washington University
St. Louis, MO 63130, USA
E-mail: maarten@aapje.wustl.edu

Santiago Peris

Grup de Física Teòrica and IFAE, Universitat Autònoma
de Barcelona, 08193 Barcelona, Spain
E-mail: peris@ifae.es

ABSTRACT: We discuss the phenomenological implications of assuming a Veneziano-type spectrum for the vector and axial-vector two-point functions in QCD at large N_c. We also compare the phenomenological results with those of Lowest-Meson Dominance, and find that they agree.

KEYWORDS: Spontaneous Symmetry Breaking, 1/N Expansion, QCD, Phenomenological Models.

JHEP01(2001)028

Contents

1. Introduction

Quantum Chromodynamics simplifies when we take the number of colors, N_c, to infinity [1]; we will denote this limit by QCD_∞. Assuming confinement, it is generally believed that the large-N_c solution qualitatively resembles the real world enough to be taken as a serious starting point for a systematic approximation by expanding in inverse powers of N_c [2].

Unfortunately, no solution to QCD_∞ has been found to date, except in the two-dimensional case [3]. We know that even in the large-N_c limit this solution has to be very complex. Large N_c, together with confinement and asymptotic freedom, predicts the existence of an infinite set of zero-width meson resonances in any channel with given quantum numbers. These resonances contribute to any Green's function made of quark bilinear operators corresponding to those quantum numbers. At very large energies a simplification occurs, as the infinite tower of resonances has to become equivalent to the free-quark loop of the parton model obtained in leading-order perturbative QCD (pQCD), with the connection provided by the operator product expansion (OPE). However, how this happens is not known, except in the two-dimensional case. We would therefore like to propose a simple model for this equivalence, which despite its simplicity is phenomenologically quite successful.

In this work we will deal only with two-point functions in the chiral limit. We limit ourselves to this choice because these are the simplest Green's functions which obey non-trivial chiral symmetry and OPE constraints. The model is based on the observation that, although the equivalence between the resonance description of QCD_∞ and the quark-gluon description of pQCD is valid at asymptotically large energy scales, phenomenology suggests that asymptotic behavior may set in already at values of $s = Q^2 \sim (1-2)\,\mathrm{GeV}^2$. In the vector channel, the ρ meson is clearly lighter than this scale s, but all higher resonances (ρ', ρ'', ρ''', etc.) appear to lie already in this asymptotic region. Therefore, it is reasonable to consider this channel as consisting of a separate ρ resonance of mass M_ρ, plus an infinite tower labeled by an index n, of masses $M_V(n)$, which corresponds to the perturbative description in a way to be described below. In the axial-vector channel, on the contrary, the a_1 may already be heavy enough to group all resonances together in a corresponding tower, $M_A(n)$. Here the exception is of course the Goldstone meson which, because of chiral-symmetry breakdown, is massless and stays away from the tower.

How these resonances are spaced as a function of the index n is not known in QCD_∞. Instead, we will rely on Regge theory [4]. It has been long suspected that there is a connection between large N_c and Regge theory [1, 2], but the precise link has never been found [5]. One of the natural features of Regge theory is the existence of so-called daughter trajectories. Their most important feature for us is that they give rise to towers of resonances with a given spin obeying an equal-spacing rule with respect to the index n, i.e. the mass squared is given by $M^2(n) = a + bn$. In two dimensions, where the solution to QCD_∞ is known, it can be shown that indeed $M^2(n) \sim n$ for large n. Here, in four dimensions, we will conjecture that this linear behavior in n is also valid for both the vector and axial-vector towers, and assume that

$$M^2_{V,A}(n) = m^2_{V,A} + n\, \Lambda^2_{V,A}\,, \tag{1.1}$$

is a reasonable approximation for all mesons except the ρ meson and the pion.

For simplicity, we will content ourselves with this equal-spacing (ES) ansatz producing only the leading term in pQCD, the parton-model logarithm. A possible remedy to this is to delay the onset of the equally spaced tower by including more and more individual resonances which are not part of the tower, and thus move the "onset" of this tower to a higher energy, where the parton-model logarithm is a better approximation. In this way the description would become more accurate. Alternatively, one could take the n dependence in eq. (1.1) to be more complicated,[1] such that non-leading perturbative corrections are reproduced from our ansatz as well. The price to pay, of course, is that this introduces more unknown parameters into our ansatz, which would have to be fixed by matching to the OPE and chiral perturbation theory (ChPT) to higher orders in a way similar to what we will describe for our simple ansatz in section 2.

[1] In principle one could also play with the n dependence of the resonances' decay constants (see below).

JHEP01(2001)028

Much of this has been said before, although perhaps not within the unifying framework of the large-N_c expansion. For instance, in the pioneering work by Bramon, Etim and Greco [6] and Sakurai [7], it was already observed that a resonance tower *including* as the first state the ρ, and obeying

$$M_V^2(n) = M_\rho^2\,(1+\alpha\;n)\,, \tag{1.2}$$

reproduces reasonably well the parton-model logarithm of the two-point vector correlation function if $\alpha = 2$ and the ρ decay constant is taken at its experimental value. This was an exciting result because the Veneziano model [8], which can be considered as the model paradigm of Regge theory, predicted exactly this value. We will later on see that, if the distribution of eq. (1.2) is assumed, the value $\alpha = 2$ is actually an exact algebraic consequence of the OPE, independent of the value of the ρ decay constant. However, in that case eq. (1.2) also leads to a negative value for the gluon condensate, a result which is clearly disfavored phenomenologically [9]. In our approach we will not take the ρ to be part of the equally spaced tower.

Also Geshkenbein considered [10] the problem of an infinite set of resonances in the vector channel and how this would match onto the OPE; this time with a general function $M_V^2(n)$. However, nobody seems to have considered an *infinite* set of resonances in the vector and axial-vector correlation functions on the same footing, together with their properties under chiral symmetry [11]. In addition, the experimental knowledge of the vector spectrum has improved considerably recently, so that an updated comparison is interesting.

In this paper, we will study the phenomenological consequences of our ansatz, in which we take equally spaced towers in both the vector and axial-vector channels with separate resonances for the ρ and the pion, to leading order in N_c, and in the chiral limit. Furthermore, although we will refer explicitly to the ρ and a_1 mesons for simplicity, it should be clear that our results also apply in the $\mathrm{U}(3)\times\mathrm{U}(3)$ chiral limit.

The rest of the paper is organized as follows. Section 2 is devoted to the presentation of our model. Section 3 contains the phenomenological consequences of this model. In section 4 we make a comparison with the Lowest-Meson Dominance approximation of ref. [12], and we conclude in section 5.

2. The model

Let us start by defining the vector two-point function as

$$\Pi^{\mu\nu}_{V,A} = i\int d^4x\, e^{iqx}\left\langle J^{\mu}_{V,A}(x) J^{\dagger\nu}_{V,A}(0)\right\rangle = \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi_{V,A}(q^2)\,, \tag{2.1}$$

with $J^\mu_V(x) = \overline{d}(x)\gamma^\mu u(x)$ and $J^\mu_A(x) = \overline{d}(x)\gamma^\mu\gamma^5 u(x)$. Both functions $\Pi_{V,A}(Q^2 \equiv -q^2)$ satisfy the dispersion relation (up to one subtraction)

$$\Pi_{V,A}(Q^2) = \int_0^\infty \frac{dt}{t+Q^2}\,\frac{1}{\pi}\,\mathrm{Im}\;\Pi_{V,A}(t)\,. \tag{2.2}$$

Following the discussion in the introduction we will assume that

$$\frac{1}{\pi}\,\mathrm{Im}\;\Pi_V(t) = 2F_\rho^2\delta\left(t-M_\rho^2\right)+2\sum_{n=0}^{\infty}F_V^2\delta\left(t-M_V^2(n)\right), \tag{2.3}$$

$$\frac{1}{\pi}\,\mathrm{Im}\;\Pi_A(t) = 2F_\pi^2\delta(t)+2\sum_{n=0}^{\infty}F_A^2\delta\left(t-M_A^2(n)\right). \tag{2.4}$$

In these expressions, F_ρ is the electromagnetic decay constant of the ρ, and $F_\pi = 93\,\mathrm{MeV}$ is the pion decay constant. F_V and $F_A = F_{a_1}$ are the corresponding constants for the towers of vector and axial-vector mesons. Note that, along with the ansatz of eq. (1.1) for $M^2_{V,A}(n)$, we take $F_{V,A}$ both to be independent of n.

Since QCD_∞ has not been solved, it is unclear what values to take for all these parameters even in the large-N_c limit. We estimate that $1/N_c$ subleading corrections are roughly of the order of $\Gamma/M \sim 20-30\%$ so that decay constants and masses are uncertain by this amount. In addition, we will give results for both $F_\pi \simeq 87\,\mathrm{MeV}$ and $F_\pi \simeq 93\,\mathrm{MeV}$ as a rough estimate of how much our results change as a consequence of chiral corrections.

The analysis of this ansatz is just a straightforward application of the Euler-Maclaurin summation formula. For instance, in the vector case one obtains

$$\sum_{n=0}^{N_V}\frac{F_V^2}{Q^2+n\Lambda_V^2+m_V^2} = \int_0^{N_V+1} dn\,\frac{F_V^2}{Q^2+n\Lambda_V^2+m_V^2}+ \\ +\frac{1}{2}\,\frac{F_V^2}{Q^2+m_V^2}+\frac{1}{12}\,\frac{F_V^2\Lambda_V^2}{(Q^2+m_V^2)^2}+\cdots, \tag{2.5}$$

where $N_V \gg 1$, with the condition that $\Lambda^2_{\mathrm{cutoff}} = (N_V+1)\Lambda_V^2+m_V^2$ is a momentum cutoff which regularizes the theory and is taken equal to the momentum cutoff in the axial channel in order to avoid spurious chiral symmetry breaking in the deep ultraviolet. Performing the integral and expanding in inverse powers of Q^2, one finds

$$\Pi_V(Q^2) = \frac{2F_V^2}{\Lambda_V^2}\log\frac{\Lambda^2_{\mathrm{cutoff}}}{Q^2}+\frac{1}{Q^2}\left[2F_\rho^2-2F_V^2\left(\frac{m_V^2}{\Lambda_V^2}-\frac{1}{2}\right)\right]+ \\ +\frac{1}{Q^4}\left[-2F_\rho^2M_\rho^2+F_V^2\Lambda_V^2\left(\frac{m_V^4}{\Lambda_V^4}-\frac{m_V^2}{\Lambda_V^2}+\frac{1}{6}\right)\right]+ \\ +\frac{1}{Q^6}\left[2F_\rho^2M_\rho^4-\frac{2}{3}F_V^2\Lambda_V^4\left(\frac{m_V^6}{\Lambda_V^6}-\frac{3}{2}\frac{m_V^4}{\Lambda_V^4}+\frac{1}{2}\frac{m_V^2}{\Lambda_V^2}\right)\right]+\mathcal{O}\left(\frac{1}{Q^8}\right), \tag{2.6}$$

and in the axial channel

$$\Pi_A(Q^2) = \frac{2F_A^2}{\Lambda_A^2}\log\frac{\Lambda^2_{\mathrm{cutoff}}}{Q^2}+\frac{1}{Q^2}\left[2F_\pi^2-2F_A^2\left(\frac{m_A^2}{\Lambda_A^2}-\frac{1}{2}\right)\right]+ \\ +\frac{1}{Q^4}\left[F_A^2\Lambda_A^2\left(\frac{m_A^4}{\Lambda_A^4}-\frac{m_A^2}{\Lambda_A^2}+\frac{1}{6}\right)\right]+ \\ +\frac{1}{Q^6}\left[-\frac{2}{3}F_A^2\Lambda_A^4\left(\frac{m_A^6}{\Lambda_A^6}-\frac{3}{2}\frac{m_A^4}{\Lambda_A^4}+\frac{1}{2}\frac{m_A^2}{\Lambda_A^2}\right)\right]+\mathcal{O}\left(\frac{1}{Q^8}\right). \tag{2.7}$$

Of course the expansions of $\Pi_{V,A}(Q^2)$ in eqs. (2.6) and (2.7) have to match their respective OPE expansions [13]. This leads to the constraints

$$\frac{2}{3}\frac{N_c}{16\pi^2} = \frac{F_V^2}{\Lambda_V^2} = \frac{F_A^2}{\Lambda_A^2}\,, \tag{2.8}$$

from the coefficient of the parton-model logarithm in front of the unit operator,

$$F_\rho^2 = F_V^2\left(\frac{m_V^2}{\Lambda_V^2} - \frac{1}{2}\right) \qquad \text{and} \qquad F_\pi^2 = F_A^2\left(\frac{m_A^2}{\Lambda_A^2} - \frac{1}{2}\right), \tag{2.9}$$

from the absence of a dimension-two operator in the OPE,

$$\begin{aligned}\frac{\alpha_s}{12\pi}\langle G_{\mu\nu}G^{\mu\nu}\rangle &= -2F_\rho^2 M_\rho^2 + F_V^2\Lambda_V^2\left(\frac{m_V^4}{\Lambda_V^4} - \frac{m_V^2}{\Lambda_V^2} + \frac{1}{6}\right)\\ &= F_A^2\Lambda_A^2\left(\frac{m_A^4}{\Lambda_A^4} - \frac{m_A^2}{\Lambda_A^2} + \frac{1}{6}\right),\end{aligned} \tag{2.10}$$

from the contribution of the gluon condensate, and

$$\begin{aligned}-\frac{28}{9}\pi\alpha_s\langle\overline{\psi}\psi\rangle^2 &= 2F_\rho^2 M_\rho^4 - \frac{2}{3}F_V^2\Lambda_V^4\left(\frac{m_V^6}{\Lambda_V^6} - \frac{3}{2}\frac{m_V^4}{\Lambda_V^4} + \frac{1}{2}\frac{m_V^2}{\Lambda_V^2}\right),\\ \frac{44}{9}\pi\alpha_s\langle\overline{\psi}\psi\rangle^2 &= -\frac{2}{3}F_A^2\Lambda_A^4\left(\frac{m_A^6}{\Lambda_A^6} - \frac{3}{2}\frac{m_V^4}{\Lambda_V^4} + \frac{1}{2}\frac{m_A^2}{\Lambda_A^2}\right),\end{aligned} \tag{2.11}$$

from the four-quark condensate.

There are two more constraints which we can impose using physical observables which depend only on $\Pi_{V,A}(Q^2)$. These are the coupling L_{10} of the $\mathcal{O}(p^4)$ chiral lagrangian [14] and the electromagnetic pion-mass difference. They are expressed in terms of $\Pi_{V,A}(Q^2)$ as

$$L_{10} = -\frac{1}{4}\frac{d}{dQ^2}\left(Q^2\Pi_{LR}(Q^2)\right)_{Q^2=0}\,, \tag{2.12}$$

and

$$m_{\pi^+} - m_{\pi^0} = -\frac{3\alpha}{8\pi m_\pi F_\pi^2}\int_0^\infty dQ^2\; Q^2\Pi_{LR}(Q^2)\,, \tag{2.13}$$

where

$$\Pi_{LR}(Q^2) \equiv \frac{1}{2}\left(\Pi_V(Q^2) - \Pi_A(Q^2)\right). \tag{2.14}$$

Given F_π, the gluon condensate and the combination $\pi\alpha_s\langle\overline{\psi}\psi\rangle^2$, we can solve for the eight parameters $F_\rho, M_\rho, F_{V,A}, m_{V,A}$ and $\Lambda_{V,A}$ from eqs. (2.8)–(2.11), and from the solution determine L_{10} and $m_{\pi^+} - m_{\pi^0}$. Thus, the system is overconstrained.

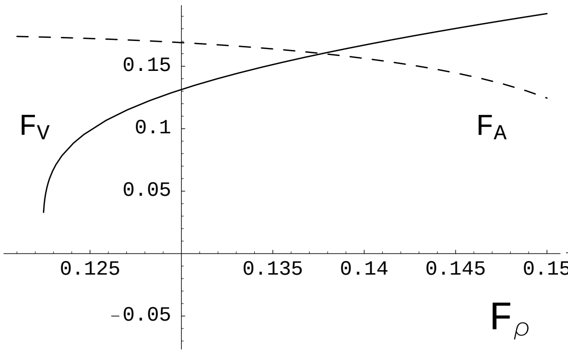

Figure 1: Decay constants F_V, F_A in eqs. (2.3) and (2.4) as a function of F_ρ in eq. (2.3). All in GeV.

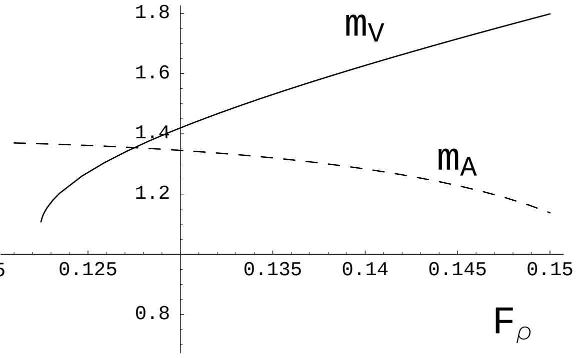

Figure 2: Resonance masses in eq. (1.1) as a function of F_ρ in eq. (2.3). All in GeV.

3. Phenomenological results

In principle, one could solve the system of eqs. (2.8)–(2.11) as indicated at the end of the previous section. However, the condensates are relatively poorly known, and we prefer to follow a different approach. We will use as inputs the values of $F_\pi = 93\,\mathrm{MeV}$ (we will also consider the value $F_\pi = 87\,\mathrm{MeV}$, which is the value in the chiral limit [14]), $M_\rho = 770\,\mathrm{MeV}$,[2] and F_ρ ranging from about 120 to 150 MeV, the latter being the experimental value. (For larger values the combination $\pi\alpha_s\langle\overline{\psi}\psi\rangle^2$ becomes negative, in contradiction with the phenomenologically most favored value. See below.) Choosing a range like this is reasonable, because we are in the large-N_c limit, and expect variations of order 20–30%. It allows us to eliminate the condensates from eqs. (2.10) and (2.11), and solve the remaining two equations along with eqs. (2.8) and (2.9) for $F_{V,A}$, $m_{V,A}$, and $\Lambda_{V,A}$, and then calculate the two condensates in terms of these.

The solutions for $F_{V,A}$ are shown in figure 1, those for $m_{V,A}$ in figure 2, and the condensates are shown in figures 3 and 4, as a function of F_ρ. Below $F_\rho \approx 123\,\mathrm{MeV}$ the equations do not have a real solution. In the axial channel, $m_A = M_{a_1} = 1230\,\mathrm{MeV}$ experimentally, while for $F_A = F_{a_1}$ a value of 135 MeV is reported in ref. [16]. F_V is not known experimentally. Phenomenological estimates for the condensates are $\alpha_s\langle G^2\rangle = 0.048\pm0.030\,\mathrm{GeV}^4$ [9], and $\pi\alpha_s\langle\overline{\psi}\psi\rangle^2 = 9\pm2\times10^{-4}\,\mathrm{GeV}^6$ [17]. Looking at figures 1 to 4, we see that very good agreement is obtained for values of F_ρ of about $130-135\,\mathrm{MeV}$. For $F_\rho = 130\,\mathrm{MeV}$ we summarize both our solution along with the experimental values in table 1, in which we also give our results for $F_\pi = 87$ instead of 93 MeV.

[2]Since one should actually use large-N_c values, one could try to vary the ρ mass as well. However, since our choice $M_\rho \simeq 770\,\mathrm{MeV}$ already does a good job, we decided to keep the analysis as simple as possible and *not* to fiddle with M_ρ, while we wait for more data to come in (for instance on the spectrum).

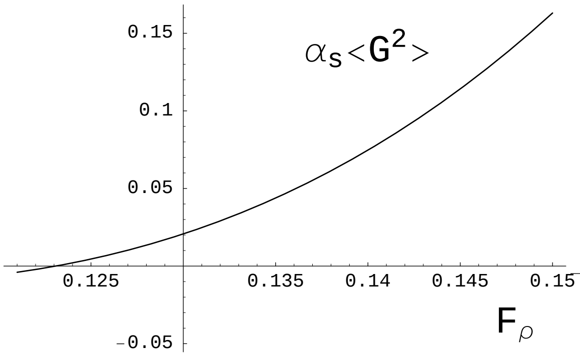

Figure 3: The gluon condensate $\alpha_s\langle G^2\rangle(\mathrm{GeV}^4)$ as a function of $F_\rho\,(\mathrm{GeV})$ in eq. (2.3).

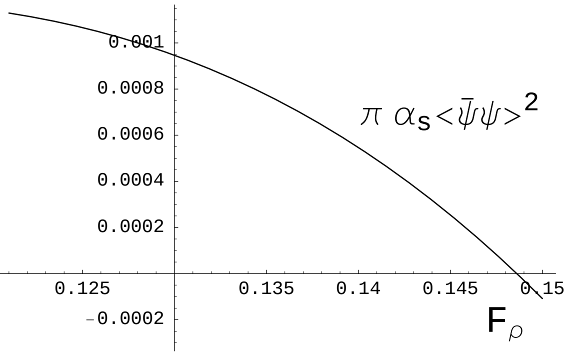

Figure 4: The four quark condensate $\pi\alpha_s\langle\overline{\psi}\psi\rangle^2\,(\mathrm{GeV}^6)$ as a function of $F_\rho\,(\mathrm{GeV})$ in eq. (2.3).

	$F_\pi = 87\,\mathrm{MeV}$	$F_\pi = 93\,\mathrm{MeV}$	Experiment
$F_\rho\,(\mathrm{MeV})$	130	130	153 ± 4 [15]
$M_\rho\,(\mathrm{GeV})$	0.77	0.77	0.7700 ± 0.0008 [15]
$F_V\,(\mathrm{MeV})$	140	130	no available data
$m_V\,(\mathrm{GeV})$	1.44	1.42	1.465 ± 0.025 [15]
$\Lambda_V\,(\mathrm{GeV})$	1.21	1.17	see table 2
$F_{a_1} = F_A\,(\mathrm{MeV})$	160	170	135 ± 30 [16]
$M_{a_1} = m_A\,(\mathrm{GeV})$	1.27	1.35	1.230 ± 0.040 [15]
$\Lambda_A\,(\mathrm{GeV})$	1.42	1.50	see table 2
$\alpha_s\langle G^2\rangle\,(\mathrm{GeV}^4)$	0.01	0.02	0.048 ± 0.030 [9]
$\pi\alpha_s\langle\overline{\psi}\psi\rangle^2(10^{-4}\,\mathrm{GeV}^6)$	7	9	(9 ± 2) [17]
$m_{\pi^+} - m_{\pi^0}\,(\mathrm{MeV})$	4.2	4.7	4.5936 ± 0.0005 [15]
$L_{10}(\mu = M_\rho)\ (10^{-3})$	-5.2	-5.6	-5.13 ± 0.19 [17]

Table 1: Results obtained for two different values of F_π. $M_\rho = 770\,\mathrm{MeV}$ is input, and we choose $F_\rho = 130\,\mathrm{MeV}$.

We may now use the values obtained for $m_{V,A}$ and $\Lambda_{V,A}$ in order to calculate the vector and axial-vector meson spectrum, from eq. (1.1). The results are shown in table 2. While experimental confirmation for the mass of the a_1' (and higher resonances) is needed, table 2 shows a good agreement between the model and the real world.

Using these results, eq. (1.1) can now be rewritten as

$$M_V^2(n) = M_{\rho'}^2\,(1 + a\ n)\,, \qquad a \simeq 0.7\,. \tag{3.1}$$

This is actually very close to the original proposal of ref. [6, 7] where the ρ is not a separate resonance from the equally spaced tower but just another member. In order to make the comparison more clear, rewrite eq. (1.2) in terms of the ρ' mass

	ρ'	ρ''	ρ'''	ρ''''	a_1	a_1'	a_1''
$F_\pi = 87\,\mathrm{MeV}$	1.4	1.9	2.2	2.5	1.3	1.9	2.4
$F_\pi = 93\,\mathrm{MeV}$	1.4	1.8	2.2	2.5	1.3	2.0	2.5
Exp. [15, 18, 19]	1.47	1.70	2.15	no data	1.23	1.64 [18], 1.8 [19]	no data

Table 2: Vector and axial-vector mass spectra in GeV with the corresponding experimental values, where available.

instead of the ρ. Equation (1.2) (with $\alpha = 2$) then reads $M_V^2(n) = M_{\rho'}^2(1+2n/3)$ and $a = 2/3$, to be compared with eq. (3.1). However, although the spectrum is quite similar, other quantities are not. As a matter of fact, note that the result $\alpha = 2$ in eq. (1.2) (i.e. $m_V^2/\Lambda_V^2 = 1/2$), can *also* be derived as a consequence of the lack of dimension-two operator in the OPE: the solution $\alpha = 2$ follows from eq. (2.9) (after setting $F_\rho = 0$, since the ρ meson belongs to the equally spaced tower of refs. [6, 7]). However, eq. (2.10) then yields a negative gluon condensate (the gluon condensate is very sensitive to variations in meson masses and decay constants).

From eq. (2.12), we find for the ES ansatz

$$L_{10} = -\frac{1}{4}\left(\frac{F_\rho^2}{M_\rho^2} + \left(\frac{m_A^2}{\Lambda_A^2} - \frac{m_V^2}{\Lambda_V^2}\right)\sum_{n=0}^{\infty}\frac{1}{(n+m_V^2/\Lambda_V^2)(n+m_A^2/\Lambda_A^2)} + \frac{2}{3}\frac{N_c}{16\pi^2}\log\frac{F_A^2}{F_V^2}\right). \tag{3.2}$$

For $F_\rho = 130\,\mathrm{MeV}$ the values we obtain are recorded in table 1. The value of L_{10} varies by less than 15% over the range of F_ρ we considered.

For $m_{\pi^+} - m_{\pi^0}$ the expression following from eq. (2.13) is rather lengthy, and we relegate it to appendix A. Again, the values for $F_\rho = 130\,\mathrm{MeV}$ are presented in table 1. In the calculation of the pion-mass difference we take $M_\pi = 135\,\mathrm{MeV}$. At $F_\rho = 150\,\mathrm{MeV}$, we find that $m_{\pi^+} - m_{\pi^0} = 3.0$ resp. $3.2\,\mathrm{MeV}$ for $F_\pi = 87$ resp. $93\,\mathrm{MeV}$. It is clear that the ES ansatz favors a lower value of F_ρ of around $130\,\mathrm{MeV}$. Since we find no solution for $F_\rho \leq 120\,\mathrm{MeV}$, the solution is quite constrained.

In principle, our model may be used to predict hitherto unknown values of quantities of phenomenological interest. Since we have restricted our attention to vector and axial-vector two-point functions, there are not many such quantities. One such quantity is a $K \to \pi$ matrix element of the electro-magnetic penguin operator Q_7,[3] which can be expressed solely in terms of Π_{LR} [20]:

$$Q_7^{3/2}(\mu) \equiv \langle \pi^+ | Q_7^{3/2} | K^+ \rangle(\mu) = \frac{3}{8\pi^2 f_\pi^2}\int_0^{\mu^2} dQ^2\, Q^4 \Pi_{LR}(Q^2)\,. \tag{3.3}$$

Even though this quantity by itself is of limited phenomenological interest, it is interesting to compare the value we obtain with that obtained from Lowest-Meson Dominance [12] (LMD; cf. section 4 below). The difference in numerical values gives

[3]The physical $K \to (\pi\pi)_{I=2}$ matrix element is related to this simpler matrix element by ChPT.

an indication of the robustness of this sum-rule approach with respect to variations in the ansatz taken for the spectral functions. In addition, lattice [21] and other phenomenological estimates [22, 24, 23] exist [25]. Note that this quantity is scale dependent, because the integral is logarithmically divergent. This is because terms up to $1/Q^4$ cancel in Π_{LR} (cf. eq. (2.14)), but not terms of order $1/Q^6$. We will return to this quantity in the next section.

4. Comparison with the Lowest-Meson Dominance approximation

It is interesting to compare the ES ansatz which we have been considering thus far with the Lowest-Meson Dominance (LMD) ansatz of ref. [12]. In the ES case, the (leading) perturbative values of $\Pi_{V,A}(Q^2)$ for large Q^2 come from the assumed Regge-like spectrum. One may also directly incorporate pQCD into an ansatz by approximating $\Pi_{V,A}(Q^2)$ with the expression one finds in pQCD which, to leading order in α_s, reads

$$\frac{1}{\pi}\,\mathrm{Im}\,\Pi_V(t)|_{\mathrm{pQCD}} = \frac{2}{3}\frac{N_{\mathrm{c}}}{16\pi^2}\,\theta\,(t - s_0)\,. \tag{4.1}$$

Here a parameter s_0 is introduced which determines the onset of the perturbative continuum, above which we expect pQCD to give accurate results. For $t \geq s_0$ this expression replaces the sums in eqs. (2.3) and (2.4). All resonances for $t < s_0$ are kept; for an illustration, see figure 5. In the simplest version of this approach, one keeps one resonance each in the vector and axial-vector channels. Since, by definition, s_0 in eq. (4.1) marks the onset of the perturbative continuum, it is taken to be the same for both vector and axial-vector channels, because chiral symmetry is not broken in perturbation theory. This ansatz has recently been shown to yield a successful description of Aleph data for the moments of the Π_{LR} function [26], it successfully predicts pseudoscalar decays [27] and, also recently, it has allowed an analytic calculation of the $K^0 - \bar{K}^0$ mixing parameter, B_K [28].

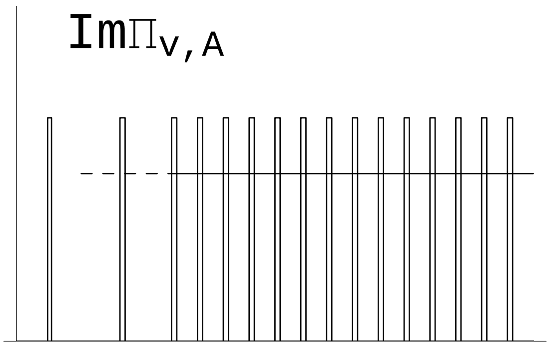

Figure 5: Schematic view of the spectrum (set of vertical lines) together with the parton model, eq. (4.1) (flat horizontal line).

Qualitatively one can understand the transition between ES and LMD in the following way. Low-energy observables are dominated by the lowest individual resonances to the extent that the effect of the continuum is suppressed by the fact that s_0 is larger than the mass (squared) of these individual resonances. For the LMD case, the continuum (4.1) produces a logarithm just like the integral in eq. (2.5) does,

	$F_\pi = 87\,\mathrm{MeV}$	$F_\pi = 93\,\mathrm{MeV}$
$F_\rho\,(\mathrm{MeV})$	122	131
$M_\rho\,(\mathrm{GeV})$	0.77	0.82
$F_{a_1}\,(\mathrm{MeV})$	87	93
$M_{a_1}\,(\mathrm{GeV})$	1.08	1.16
$\alpha_s\langle G^2\rangle\,(\mathrm{GeV}^4)$	0.01	0.02
$\pi\alpha_s\langle\overline{\psi}\psi\rangle^2(10^{-4}\,\mathrm{GeV}^6)$	13	19
$m_{\pi^+} - m_{\pi^0}\,(\mathrm{MeV})$	5.2	6.0
$L_{10}(\mu = M_\rho)\ (10^{-3})$	-4.8	-4.8

Table 3: LMD results obtained for two different values of F_π.

with $s_0 = 24\pi^2 F_\rho^2/N_\mathrm{c} = m_V^2$ (see refs. [12, 29] for details on the LMD ansatz). Taking also into account the leading correction term in eq. (2.5) to order $1/Q^2$, we find $s_0 = m_V^2\left(1 - \frac{1}{2}\Lambda_V^2/m_V^2\right)$. Taking Λ_V^2/m_V^2 smaller (keeping F_V^2/Λ_V^2 fixed, cf. eq. (2.8)) clearly approaches the perturbative approximation of eq. (4.1), with $s_0 \to m_V^2$, and with the correction terms on the r.h.s. of eq. (2.5) becoming smaller (for any value of Q^2). In this sense, the LMD approach can be seen as following from the ES approach in the limit $\Lambda_V^2/m_V^2 \to 0$, at least in the vector channel. (In the axial channel, a similar argument would apply if we had taken one separate resonance apart from the tower in the ES approach. In our case, m_A is relatively small, and $\frac{1}{2}\Lambda_A^2/m_A^2$ is substantially bigger than $\frac{1}{2}\Lambda_V^2/m_V^2$.)

Of course, it is easy to put α_s corrections into eq. (4.1) in the LMD approach, while this is less straightforward in the ES approach (it requires a more complicated n dependence of $M_{V,A}^2$ and $F_{V,A}^2$). One can also improve the extent to which LMD approximates ES by keeping more individual resonances explicit in the LMD ansatz. This will move up s_0 or equivalently m_V^2, hence reducing the size of Λ_V^2/m_V^2.

Using the same information from the OPE, *and* the relation $F_\rho^2 = 2F_\pi^2$ [30], one finds that in the LMD approach s_0 is determined to be $s_0 = (4\pi F_\pi)^2$ from the vector channel, and then, from the axial-vector channel, that M_{a_1} is close to, but smaller than $\sqrt{s_0}$ [29].[4] This implies that within LMD the a_1 resonance has to be kept, and cannot be folded into the perturbative continuum, eq. (4.1). Higher resonances are well above $\sqrt{s_0}$ (whether we take their experimental values, or those predicted from ES, cf. table 2). We will therefore compare the phenomenology of the ES ansatz, summarized in tables 1 and 2, with that of the LMD ansatz with one explicit resonance in each channel (in addition to the pion). The LMD results are presented in table 3. All quantities displayed in table 3 are predicted in terms of F_π with the LMD ansatz [12, 29], using the relation $F_\rho^2 = 2F_\pi^2$, but (obviously) no predictions are made for excited states in both channels in this case. In the comparison with the

[4]In comparing with ref. [29], note that $F_\rho = f_\rho M_\rho$, etc.

results in table 1, one should keep in mind that the relation $F_\rho^2 = 2F_\pi^2$ was not used with the ES ansatz. However, for $F_\pi = 87$, resp. $93\,\text{MeV}$, $F_\rho = 130\,\text{MeV}$ satisfies this relation at the 10, resp. 2% level.

Comparing the results of ES and LMD, it is clear that both approaches yield consistent results, especially if one takes into account that all results are in the large N_c and chiral limits, so that deviations of less than 30% are not significant. Possible exceptions to this are F_{a_1} and $\pi\alpha_s\langle\overline{\psi}\psi\rangle^2$, for which the LMD values are smaller, resp. larger, than the ES value. Unfortunately, the experimental values, while closer to the ES values, are rather poorly known. Possibly with these two exceptions, the differences between tables 1 and 3 give a good indication of the spread one can expect when one combines QCD techniques such as large N_c, OPE and pQCD with an ansatz for the spectrum in order to obtain explicit solutions.

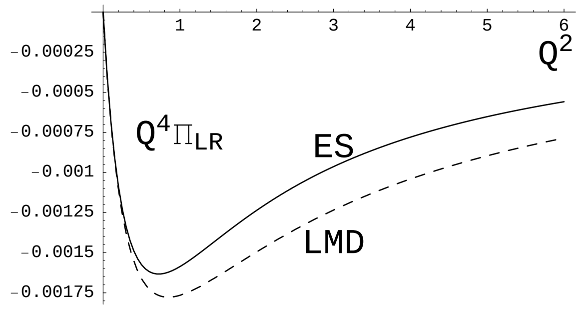

Figure 6: The function $Q^4\Pi_{LR}(Q^2)(\text{GeV}^4)$ for our equal-spacing model (ES) (solid line) and its lowest meson dominance version (LMD) (dashed line) as a function of the euclidean momentum Q^2 (GeV^2).

Finally, we compare the ES prediction for $Q_7^{3/2}(\mu)$ with the LMD prediction for the same number. First, we show the function $Q^4\Pi_{LR}(Q^2)$ in figure 6, for both the ES and LMD ansätze. We evaluated the ES value of $Q_7^{3/2}(\mu)$ at $\mu = 2\,\text{GeV}$ from the area under the curve using μ as a sharp cutoff (choosing $F_\rho = 130\,\text{MeV}$ and $M_\rho = 770\,\text{MeV}$), and quote the LMD value from ref. [20], finding,

$$Q_7^{3/2}(\mu = 2\,\text{GeV}) = \begin{cases} -0.021\,(\text{GeV})^4\,, & \text{ES}, \quad F_\pi = 93\,\text{MeV}, \\ -0.019\,(\text{GeV})^4\,, & \text{ES}, \quad F_\pi = 87\,\text{MeV}, \\ -0.024\,(\text{GeV})^4\,, & \text{LMD}\,. \end{cases} \tag{4.2}$$

We took again $F_\rho^2 = 2F_\pi^2$ in the LMD case, in which case the LMD value is independent of F_π (for a fixed M_ρ). We see that, as in the case of most other quantities, the predicted value we obtain for $Q_7^{3/2}(\mu = 2\,\text{GeV})$ is quite robust to the variation of the ansatz for the vector and axial-vector spectral functions we have been considering. The values we obtain are fully consistent with each other within the context of the large N_c and chiral limits.

5. Conclusion

In this paper we have considered the vector and axial-vector current two-point functions in QCD. The combination of taking the large N_c and chiral limits, the use of asymptotic freedom and the OPE constrain the possible form of these two-point functions. However, this is not enough to obtain concrete predictions: more knowledge, or lacking that, an ansatz for the spectral functions in these two channels is still needed.

A particular ansatz which has been considered in the literature is the LMD ansatz, in which the high-energy part of the spectral functions is calculated in pQCD, while one (or a few) explicit resonances are kept at low to medium energy. This approach has been quite successful in reproducing known phenomenology, and should therefore be useful in estimating as yet unknown hadronic quantities, such as weak matrix elements. However, although LMD yields a rather intuitive picture it is not a simple matter to give it a fully systematic treatment. Such systematic treatment should, in particular, produce a theory of the so-called quark-hadron duality violations [31], a problem which is clearly beyond the scope of the present work. Consequently, determinations of the systematic errors associated with this approach have to rely at some point on phenomenological experience.

One way to address this issue is to vary the nature of the ansatz within the confines of our general picture of the strong interactions, and this is what we have done in this paper. Instead of LMD, we have based our ansatz on Regge theory, except for the lowest states in each channel (the ρ meson and the pion). For the precise form of the ES ansatz, see eqs. (2.3) and (2.4). The hadron phenomenology calculated with this assumption is again very good, and agrees well with what is found with LMD (cf. tables 1 and 3). Possible exceptions perhaps are F_{a_1} and $\pi\alpha_s\langle\overline{\psi}\psi\rangle^2$; experimentally these quantities are not that well known. In general, because of the use of large N_c and chiral limits, agreement at the 20-30% level, between ES and LMD, or with experiment, should be considered satisfactory.

The gluon and four-quark condensates are rather sensitive to the values of other quantities (masses and decay constants). Within our solution (cf. table 1), we find that the ρ comes out to be very close to lying on the daughter trajectory predicted by Regge theory in that $M_{\rho'}^2 \approx M_\rho^2 + \Lambda_V^2$ and $F_\rho \approx F_V$. However, requiring the ρ to be *precisely* on this trajectory leads to a negative gluon condensate. Since large N_c (supplemented with Regge theory) and pQCD suggest that asymptotically the tower of resonances has to be equally spaced, it makes intuitive sense to model the resonance spectrum with an equally spaced tower above 1 GeV, keeping those below 1 GeV separate. The ES ansatz goes beyond LMD in that it yields a value for the spacing between the resonances (the parameters $\Lambda_{V,A}$), and leads to a reasonable resonance pattern in the vector and axial-vector channels (cf. table 2).

We have then used the ES ansatz in order to predict a weak matrix element, $Q_7^{3/2}(\mu)$, which is of interest in the context of non-leptonic kaon decays. We find a value rather close to that predicted from LMD, thus lending support to the credibility of this prediction.

Finally, we discussed the extent to which the two ansätze are equivalent. We qualitatively argued that in the limit in which we take the spacing between the resonances in the ES tower to zero one recovers the LMD approach. However, with LMD, we kept both the ρ and the a_1 as separate resonances (because one finds that the perturbative threshold s_0 is larger than $M^2_{a_1}$), while we did take the a_1 to be the first resonance of the ES tower.

After all these years we still do not have a real understanding of the large-N_c expansion in QCD. Clearly the ES model is not meant to be a substitute for large-N_c QCD at the theoretical level but, given its phenomenological success, it may help in our understanding of this limit of QCD.

Acknowledgments

MG and SP thank E. de Rafael for useful discussions and M. Knecht for reading the manuscript. SP thanks A. Bramon for interesting conversations on Regge theory and F. Jegerlehner for drawing his attention to ref. [10] in the course of this work. MG and SP thank the IFAE at the Universitat Autònoma de Barcelona and the Department of Physics at Washington University for mutual hospitality. MG was supported in part by the US Dept. of Energy and by a fellowship of the Spanish Government SAB1998-0171. SP was supported by CICYT-AEN99-0766 and by TMR, EC-Contract No. ERBFMRX-CT980169 (EuroDaphne).

A. Electromagnetic pion mass difference

The expression for the electromagnetic pion mass difference, following from the ES ansatz follows from eq. (2.13), is

$$
\begin{aligned}
m_{\pi^+} - m_{\pi^0} = \frac{3\alpha}{8\pi m_\pi f_\pi^2} \Bigg\{ & -\frac{1}{2} F_\rho^2 M_\rho^2 + \frac{1}{2} F_A^2 m_A^2 - \\
& - \frac{N_c}{24\pi^2} \left[\frac{11}{12} m_V^4 \log \frac{M_\rho^2}{m_V^2} - \frac{7}{12} \left(m_V^2 + \Lambda_V^2\right)^2 \log \frac{M_\rho^2}{m_V^2 + \Lambda_V^2} + \right. \\
& \qquad \left. + \frac{1}{6} \left(m_V^2 + 2\Lambda_V^2\right)^2 \log \frac{M_\rho^2}{m_V^2 + 2\Lambda_V^2} \right] + \\
& + \frac{N_c}{24\pi^2} \left[\frac{11}{12} \left(m_A^2 + \Lambda_A^2\right)^2 \log \frac{m_A^2}{m_A^2 + \Lambda_A^2} - \right. \\
& \qquad - \frac{7}{12} \left(m_A^2 + 2\Lambda_A^2\right)^2 \log \frac{m_A^2}{m_A^2 + 2\Lambda_A^2} + \\
& \qquad \left. + \frac{1}{6} \left(m_A^2 + 3\Lambda_A^2\right)^2 \log \frac{m_A^2}{m_A^2 + 3\Lambda_A^2} \right] + \\
& + \frac{1}{2} \left[2F_\rho^2 M_\rho^2 - \frac{24\pi^2}{N_c} F_\rho^4 + \frac{1}{12} F_V^2 \Lambda_V^2 \right] \log \frac{m_A^2}{M_\rho^2} - \\
& - 3F_V^2 \Lambda_V^2 G\left(\frac{m_V^2}{\Lambda_V^2}\right) + 3F_A^2 \Lambda_A^2 G\left(\frac{m_A^2}{\Lambda_A^2}\right) \Bigg\}, \qquad \text{(A.1)}
\end{aligned}
$$

JHEP01(2001)028

where

$$G(x)=\sum_{n=0}^{\infty}\left(\frac{1}{(n+x)^2}+4(n+1+x)^3\log\frac{n+x}{n+1+x}-\right.$$
$$-\ 3(n+2+x)^3\log\frac{n+x}{n+2+x}+\frac{4}{3}(n+3+x)^3\log\frac{n+x}{n+3+x}-$$
$$\left.-\ \frac{1}{4}(n+4+x)^3\log\frac{n+x}{n+4+x}\right). \tag{A.2}$$

This expression results from calculating the integral in eq. (2.13), which is finite, because $\Pi_{LR}(Q^2)\sim 4\pi\alpha_s\langle\overline{\psi}\psi\rangle^2/Q^6$ for large Q^2. By partial integration

$$\int_0^\infty dQ^2\,Q^2\Pi_{LR}(Q^2)=-\frac{1}{2}\int_0^\infty dQ^2\,Q^4\frac{d}{dQ^2}\Pi_{LR}(Q^2)\,. \tag{A.3}$$

The sums over n inside $\frac{d}{dQ^2}\Pi_{LR}(Q^2)$ now converge. Next, one would like to make use of the fact that terms up to order $1/Q^8$ in $\frac{d}{dQ^2}\Pi_{LR}(Q^2)$ vanish, as follows from eqs. (2.8)–(2.10). This can be done by writing

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{F_V^2}{(Q^2+m_V^2+n\Lambda_V^2)^2}&=\sum_{n=0}^{\infty}\left(\frac{F_V^2}{(Q^2+m_V^2+n\Lambda_V^2)(Q^2+m_V^2+(n+1)\Lambda_V^2)}+\right.\\&\qquad\left.+\ \frac{F_V^2}{(Q^2+m_V^2+n\Lambda_V^2)^2(Q^2+m_V^2+(n+1)\Lambda_V^2)}\right)\\&=\frac{F_V^2}{\Lambda_V^2(Q^2+m_V^2)}+\\&\quad+\sum_{n=0}^{\infty}\frac{F_V^2}{(Q^2+m_V^2+n\Lambda_V^2)^2(Q^2+m_V^2+(n+1)\Lambda_V^2)}\,,\end{aligned} \tag{A.4}$$

using

$$\begin{aligned}&\sum_{n=0}^{N}\frac{1}{(p+nq)(p+(n+1)q)\cdots(p+(n+\ell)q)}=\\&\quad=\frac{1}{\ell q}\left(\frac{1}{p(p+q)\cdots(p+(\ell-1)q)}-\frac{1}{(p+(N+1)q)\cdots(p+(N+\ell)q)}\right)\end{aligned} \tag{A.5}$$

with $N=\infty$ and $\ell=1$. This process can be repeated by employing this equation for higher values of ℓ until the "left-over" sum on the r.h.s. of eq. (A.4) is of order $1/Q^8$, so that the integral over Q^2 in eq. (A.3) can be done term by term. The remaining terms combine with similar terms in the axial channel, and the Q^2 integral over these combined terms is finite as well, by virtue of eqs. (2.8)–(2.10). One ends up with the expression given in eqs. (A.1) and (A.2).

References

[1] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[2] E. Witten, *Baryons in the* $1/N$ *expansion*, *Nucl. Phys.* **B 160** (1979) 57;
See also A.V. Manohar, *large-N QCD*, *Probing the standard model of particle interactions*, F. David and R. Gupta eds., Les Houches Summer School in Particle Physics, July 1997 [`hep-ph/9802419`].

[3] G. 't Hooft, *A two-dimensional model for mesons*, *Nucl. Phys.* **B 75** (1974) 461;
C.G. Callan Jr., N. Coote and D.J. Gross, *Two-dimensional Yang-Mills theory: a model of quark confinement*, *Phys. Rev.* **D 13** (1976) 1649;
M.B. Einhorn, *Form-factors and deep inelastic scattering in two-dimensional quantum chromodynamics*, *Phys. Rev.* **D 14** (1976) 3451.

[4] P.D.B. Collins, *Phys. Rep.* **C1** (1971) 103.

[5] For a very recent proposal in this direction see A.D. Patel, A.D. Patel, *A derivation of Regge trajectories in large-N transverse lattice QCD*, `hep-lat/0012004` presented at *Lattice 2000*, Bangalore, India.

[6] A. Bramon, E. Etim and M. Greco, *A vector meson dominance approach to scale invariance*, *Phys. Lett.* **B 41** (1972) 609;
M. Greco, *Deep inelastic processes*, *Nucl. Phys.* **B 63** (1973) 398.

[7] J.J. Sakurai, *Duality in* $e^+e^- \to$ *hadrons?*, *Phys. Lett.* **B 46** (1973) 207.

[8] G. Veneziano, *Construction of a crossing-symmetric, Regge behaved amplitude for linearly rising trajectories*, *Nuovo Cim.* **A57** (1968) 190.

[9] See for instance, F.J. Yndurain, *Gluon condensate from superconvergent QCD sum rule*, *Phys. Rep.* **320** (1999) 287 [`hep-ph/9903457`].

[10] B.V. Geshkenbein, *Quantum chromodynamics with infinite number of vector mesons*, *Sov. J. Nucl. Phys.* **49** (1989) 705.

[11] For a very related analysis in which, however, a *finite* number of resonances is considered, see M. Knecht and E. de Rafael, *Patterns of spontaneous chiral symmetry breaking in the large-*N_c *limit of QCD-like theories*, *Phys. Lett.* **B 424** (1998) 335 [`hep-ph/9712457`].

[12] S. Peris, M. Perrottet and E. de Rafael, *Matching long and short distances in large-N(c) QCD*, *J. High Energy Phys.* **05** (1998) 011 [`hep-ph/9805442`];
M. Knecht, S. Peris and E. de Rafael, *A new approach to weak amplitudes in large-*N_c *QCD*, *Nucl. Phys.* **86** *(Proc. Suppl.)* (2000) 279 [`hep-ph/9910396`].

[13] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *QCD and resonance physics. sum rules*, *Nucl. Phys.* **B 147** (1979) 385; *QCD and resonance physics: applications*, *Nucl. Phys.* **B 147** (1979) 448.

[14] J. Gasser and H. Leutwyler, *Chiral perturbation theory: expansions in the mass of the strange quark*, *Nucl. Phys.* **B 250** (1985) 465.

[15] Particle Data Group, *Review of particle physics Eur. Phys. J.* **C 15** (2000) 1.

[16] G. Ecker, J. Gasser, A. Pich and E. de Rafael, *The role of resonances in chiral perturbation theory*, *Nucl. Phys.* **B 321** (1989) 311.

[17] M. Davier, L. Girlanda, A. Hocker and J. Stern, *Finite energy chiral sum rules and tau spectral functions*, *Phys. Rev.* **D 58** (1998) 096014 [hep-ph/9802447].

[18] C.A. Baker et al., *Evidence for a $J_{pc} = 1 + + \ i = 1$ meson at 1640 MeV*, *Phys. Lett.* **B 449** (1999) 114.

[19] VES collaboration, *New results from VES*, hep-ex/9905002.

[20] M. Knecht, S. Peris and E. de Rafael, *Matrix elements of electroweak penguin operators in the $1/N_c$ expansion*, *Phys. Lett.* **B 457** (1999) 227 [hep-ph/9812471]; *A new approach to weak amplitudes in large-N_c QCD*, *Nucl. Phys.* **86** *(Proc. Suppl.)* (2000) 279 [hep-ph/9910396].

[21] A. Donini, V. Gimenez, L. Giusti and G. Martinelli, *Renormalization group invariant matrix elements of $\Delta s = 2$ and $\Delta I = 3/2$ four-fermion operators without quark masses*, *Phys. Lett.* **B 470** (1999) 233 [hep-lat/9910017];
R. Gupta, *Quark masses, B-parameters and CP-violation parameters ε and ε'/ε*, in *Physics of mass*, Proceedings of International Conference Orbis Scientiae 1997 II, Miami Beach, Florida, 1997 B. Kursunoglu et al. eds., [hep-ph/9801412] and references therein.

[22] J.F. Donoghue and E. Golowich, *Dispersive calculation of $B_7^{3/2}$ and $B_8^{3/2}$ in the chiral limit*, *Phys. Lett.* **B 478** (2000) 172 [hep-ph/9911309].

[23] S. Narison, *New QCD-estimate of the kaon penguin matrix elements and ε'/ε*, *Nucl. Phys.* **B 593** (2001) 3 [hep-ph/0004247].

[24] J. Bijnens and J. Prades, *ϵ'_K/ϵ_K in the chiral limit*, *J. High Energy Phys.* **06** (2000) 035 [hep-ph/0005189].

[25] A critical comparison between the different estimates is underway: M. Knecht, S. Peris and E. de Rafael, in preparation.

[26] S. Peris, B. Phily and E. de Rafael, *Tests of large-N_c QCD from hadronic tau decay*, *Phys. Rev. Lett.* **86** (2001) 14 [hep-ph/0007338];
E. de Rafael, *Minimal hadronic ansatz to large-N_c QCD and hadronic tau decay*, hep-ph/0010209;
S. Narison, *V-A hadronic tau decays: a QCD laboratory*, hep-ph/0012019.

[27] M. Knecht, S. Peris, M. Perrottet and E. de Rafael, *Decay of pseudoscalars into lepton pairs and large-N_c QCD*, *Phys. Rev. Lett.* **83** (1999) 5230 [hep-ph/9908283].

[28] S. Peris and E. de Rafael, *$K^0\bar{K}^0$ mixing in the $1/N_c$ expansion*, *Phys. Lett.* **B 490** (2000) 213 [hep-ph/0006146];
S. Peris, *B_K in the chiral limit within the $1/N_c$ expansion*, hep-ph/0010162 presented at the *International euroconference in quantum chromodynamics: 15 years of QCD*, Montpellier, France, July 2000.

[29] M.F.L. Golterman and S. Peris, *The 7/11 rule: an estimate of m_ρ/f_π*, *Phys. Rev.* **D 61** (2000) 034018 [hep-ph/9908252].

[30] G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, *Chiral lagrangians for massive spin 1 fields*, *Phys. Lett.* **B 223** (1989) 425.

[31] M. Shifman, *Quark-hadron duality*, hep-ph/0009131, to be published in *At the frontier of particle physics/handbook of QCD*, M. Shifman ed., World Scientific, 2001.